Euler's equation in 1-D for Compressible inviscid flows

Conservation mass: $\rho_t + (\rho u)_x = 0$ (1.1)

" momentum: $\boxed{(\rho u)_t + \left(\frac{(\rho u)^2}{2\rho} + p\right)_x = 0}$ (1.2)

" energy : $e_t + \left[(e+p)\frac{m}{\rho}\right]_x = 0$ (1.3)

Defining

$$\vec{V} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} \quad \text{and} \quad \vec{f}(\vec{V}) = \begin{bmatrix} \rho u \\ \frac{(\rho u)^2}{\rho} + p \\ (e+p)\frac{m}{\rho} \end{bmatrix}$$

The System of PDE equs. given by (1.1) - (1.3) Can be written as

$$\boxed{\vec{V}_t + (f(\vec{V}))_x = \vec{0}} \quad (1.4)$$

A weaker version of (1.4) can be expressed as a $\boxed{\text{Conservation Law}}$:

$$\boxed{\frac{d}{dt}\int_\alpha^\beta \vec{V}(x,t)\,dx = -\vec{f}(\vec{V}(\beta,t)) + \vec{f}(\vec{V}(\alpha,t))} \quad (1.5)$$

If we impose additional conditions like $\rho \equiv$ constant and $p \equiv$ constant in (1.2), it reduces to

$$\cancel{\rho} u_t + \left(\cancel{\rho} \frac{u^2}{2}\right)_x = 0$$

or

$$\boxed{u_t + u u_x = 0} \qquad (2.1)$$

This equation is known as inviscid incompressible Burger's equation.

It can also be obtained from our previous incompressible and inviscid fluid equations.

$$\boxed{\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = A_e - \frac{\nabla p}{\rho}} \qquad (2.2)$$

If No external forces: $A_e = 0$. Also, if p const. $\nabla p = 0$ and (2.2) reduces to

$$\boxed{\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = \vec{0}} \qquad (2.3)$$

If we only consider 1-D and $\vec{u} = (u, 0, 0)$ we obtain

$$\boxed{u_t + u u_x = 0}$$

$u_1 = u$

Recall: $(\vec{u} \cdot \nabla)\vec{u} = \left(u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}\right)(u_1, u_2, u_3) = \vec{u} \cdot \nabla u_1 + \dots$

Defining $\quad f(u) = \frac{u^2}{2}$

Then $\quad \left[f(u(x,t))\right]_x = \frac{\cancel{2}\, u(x,t)}{\cancel{2}}\, u_x(x,t)$

$$= (u\,u_x)(x,t)$$

then (2.1) as a conservation Law is given by

$$\boxed{\frac{d}{dt}\int_{\alpha}^{\beta} u(x,t)\,dx = \left[-f(u(x,t))\right]_{\alpha}^{\beta}} \qquad (3.1)$$

$$\begin{cases} u_t + u u_x = 0 & (4.1) \\ u(x,0) = f(x) & (4.2 \end{cases}$$

Along $\mathscr{C}$: $\hat{u}(t) = u(x(t),t)$ (4.3)

$$\frac{d\hat{u}}{dt}(t) = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x}$$

I If $\boxed{\frac{dx}{dt}(t) = u(x(t),t)}$ (4.4) along $\mathscr{C}$.

$$\Rightarrow \quad \frac{d\hat{u}}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}\left[u(x(t),t)\right] = \frac{d\hat{u}}{dt} = 0.$$

$\Rightarrow$ II $\boxed{\hat{u}(t) = u(x(t),t) = K}$ const along $\mathscr{C}$.

and using I.C.

$$\hat{u}(0) = u(\overset{x_0}{x(0)},0) = K \quad (4.5)$$

$$= f(x_0)$$

and $\boxed{u(x(t),t) = u(x_0,0) = K.}$ along $\mathscr{C}$.

Therefore, $\frac{dx}{dt}(t) = u(x_0,0) \overset{I.C.}{=} f(x_0)$

$$\Rightarrow \boxed{x(t) = f(x_0)t + d} \text{ or } \boxed{x(t) = u(x_0,0)t + x_0}$$

$$\Rightarrow \quad x_0 = x(0) = d \Rightarrow \boxed{d = x_0}$$

III

$$\boxed{x(t) = f(x_0)t + x_0} \Rightarrow \boxed{x_0 = x - f(x_0)t} \quad (4.6)$$

So, $$\boxed{u(x(t),t) = u(x(0),0) = f(x_0)} \quad (5.1)$$

Substituting x_0 by $x - f(x_0)t$ according to (4.6)

we are led to

$$\boxed{u(x(t),t) = f\left(x - f(x_0)t\right)} \text{ along } \mathcal{C} \quad (5.2)$$

and because the arbitrariness of (x,t)

$$\boxed{u(x,t) = f\left(x - f(x_0)t\right)}$$

The way to evaluate this function (implicit) is:

1) Given (x,t) find if possible analytically the x_0 corresponding to it by solving $\underline{x - f(x_0)t = x_0}$ for x_0

2) Once x_0 is known for the given pair (x,t), evaluate $f(x_0)$

3) Then, $\boxed{u(x,t) = f\left(x - f(x_0)t\right) = f(x_0)}$

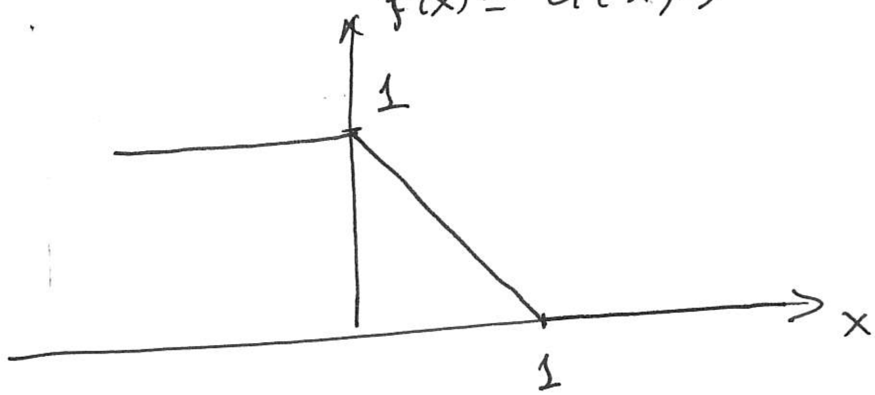
Application: Consider the IVP:

$$\begin{cases} u_t + u u_x = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) = \begin{cases} 1, & x < 0 \\ 1-x, & 0 \le x < 1 \\ 0, & x \geq 1 \end{cases} \end{cases}$$

The soln:

$$u(x,t) = f\left(x - f(x_0)t\right)$$

$$f(x) = u(x,0)$$

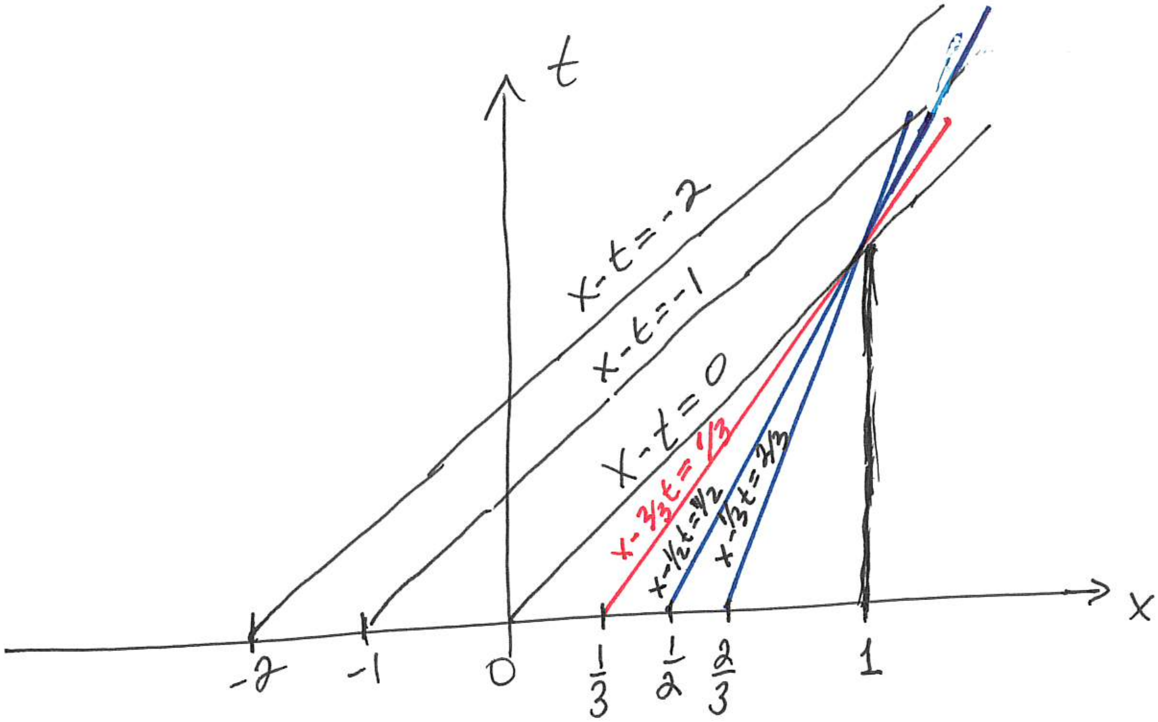


The Characteristics are drawn in page 7

$$\boxed{x - f(x_0)t = x_0}$$

a) If $x_0 < 0$, then $f(x_0) = 1$ and the corresponding charcts: $\boxed{x - t = x_0}$

b) If $x_0 > 1$, then $f(x_0) = 0$ and charact. $\boxed{x = x_0}$



Charact passing by $(1/3, 0)$

$f(1/3) = 1 - 1/3 = 2/3 \Rightarrow$ x-t plane: $t = \dfrac{x - x_0}{f(x_0)}$

slope $= \dfrac{1}{f(x_0)}$.

In particular, -plane -plane

at $x_0 = 1/3$ slope $= \dfrac{3}{2}$, slope $f(x_0) = 2/3$

t-x plane at $x_0 = 1/2$ slope $= 2$, slope $f(x_0) = 1/2$

at $x_0 = \dfrac{2}{3}$ slope $= 3$, slope $f(x_0) = 1/3$

c) If $0<x_0<1$, then $f(x_0) = 1 - x_0$.

and corresp. charact. is

$$\boxed{x - (1-x_0)t = x_0}$$

For example, if

$x_0 = 1/2$, $\quad \boxed{x - \frac{t}{2} = x_0 = 1/2}$

or $\quad \boxed{x(t) = \frac{t+1}{2}}$

A pair (x,t) is in the charact.

$$x - (1-x_0)t = x_0, \text{ and } \quad 0<x_0<1.$$

if and only if $x - t = x_0 - x_0 t = x_0(1-t)$

Then $\quad \boxed{x_0 = \frac{x-t}{1-t}}, \quad \frac{0<x_0<1.}{t \neq 1}$

or $\quad \boxed{0 < \frac{x-t}{1-t} < 1}$

Also, $\quad f(x_0) = 1 - x_0 = 1 - \frac{x-t}{1-t} = \frac{1 - \cancel{t} - x + \cancel{t}}{1-t}$

or $\quad \boxed{f(x_0) = \frac{1-x}{1-t}}$

Two cases arise:

(I) $\underline{\underline{t<1}}$

$$u(x,t)=\begin{cases} 1, & t<1 \text{ and } x-t\le 0 \\ \dfrac{1-x}{1-t}, & t<1 \text{ and } 0<\dfrac{x-t}{1-t}<1 \\ 0, & t<1 \text{ and } x>0 \end{cases}$$

(II) If $t>1$ we will show that

$$u(x,t)=\begin{cases} 1, & x-t/2<1/2 \\ 0, & x-t/2\geq 1/2 \end{cases}$$

For $t>1$, we have a multivalued function. We still can have a univalued but discontinuous function along a smooth curve $x = \xi(t)$, $t>1$

The problem is that the PDE: $u_t + u u_x = 0$ will not be satisfied for this function

So, we consider the integral form of the Conservation Law: (3.1):

$\alpha < \xi(t) < \beta$ for all $t>0$.

$$\frac{d}{dt}\int_{\alpha}^{\beta} u(x(t),t)\,dx = -f(u(\beta,t)) + f(u(\alpha,t))$$

$$= f(u(x,t))\Big|_{x=\alpha}^{x=\beta} \quad (10.1)$$

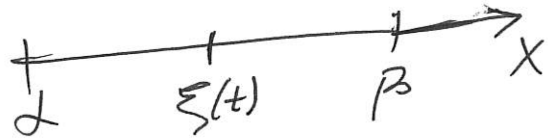
For Burger's eqn.

$$u_t + u u_x = 0, \quad \boxed{f(u) = \frac{u^2}{2}}$$

Since $u_t + (f(u))_x = u_t + \left(\frac{u^2}{2}\right)_x$

$$= u_t + u u_x .$$

We separate the integral in (10.1) using the schock location as an intermediate point



$$\frac{d}{dt}\int_{\alpha}^{\beta} u(x(t),t)\,dx = \frac{d}{dt}\int_{\alpha}^{\xi^-=\xi(t^-)} u(x(t),t)\,dx + \int_{\xi^+=\xi(t^+)}^{\beta} u(x(t),t)\,dx$$

$$= -f(u(x,t))\Big|_{\alpha}^{\beta} \qquad (11.1)$$

Applying Leibniz's rule

$$\int_{\alpha}^{\xi^-} \frac{\partial u}{\partial t}(x,t)\,dx + u(\xi^-,t)\,\dot{\xi}^-(t)$$

$$+ \int_{\xi^+}^{\beta} \frac{\partial u}{\partial t}(x,t)\,dx - u(\xi^+,t)\,\dot{\xi}^+(t) = -f(u(x,t))\Big|_{\alpha}^{\beta}$$

The PDE is valid in the intervals (α,ξ^-) and (ξ^+,β)

then,

$$\int_{\alpha}^{\xi^-} -[f(u(x,t))]_x\,dx + u^-\dot{\xi}^- + \int_{\xi^+}^{\beta} -[f(u(x,t))]_x\,dx$$

$$- u^+\dot{\xi}^+ = -f(u(x,t))\Big|_{\alpha}^{\beta} \qquad (11.2)$$

Applying Fund. Thm of Calculus to (11.2), we arrive to

$$-f(u^-) + \cancel{f(u(\alpha,t))} + u^-\dot{\xi}^- - \cancel{f(u(\beta,t))}$$

$$+ f(u^+) - u^+\dot{\xi}^+ = -f(u(x,t))\Big|_\alpha^\beta$$

$$= -\cancel{f(u(\beta,t))} + \cancel{f(u(\alpha,t))}$$

Now, $\xi(t)$ is smooth then, $\dot{\xi}^+ = \dot{\xi}^- = \dot{\xi}(t)$

And we arrive to

$$\dot{\xi}(t)\left(u^- - u^+\right) = f(u^-) - f(u^+)$$

or

$$\boxed{\dot{\xi}(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}}$$

Known as Rankine-Hugoniot jump condition.

Now, $f(u^+) = \frac{(u^+)^2}{2} = 0$, $u^+=0,\ u^-=1$ Values at the left and right of schock.

$f(u^-) = \frac{(u^-)^2}{2} = \frac{1}{2}$

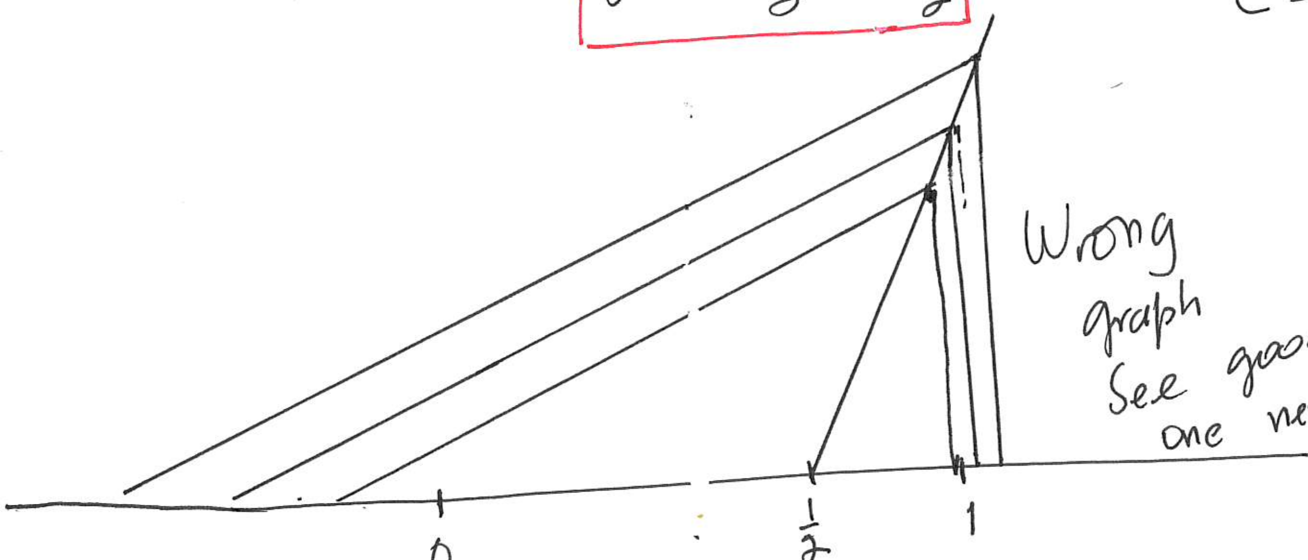
Then,

$$\dot{\xi}(t) = \frac{-1/2}{0-1} = 1/2$$

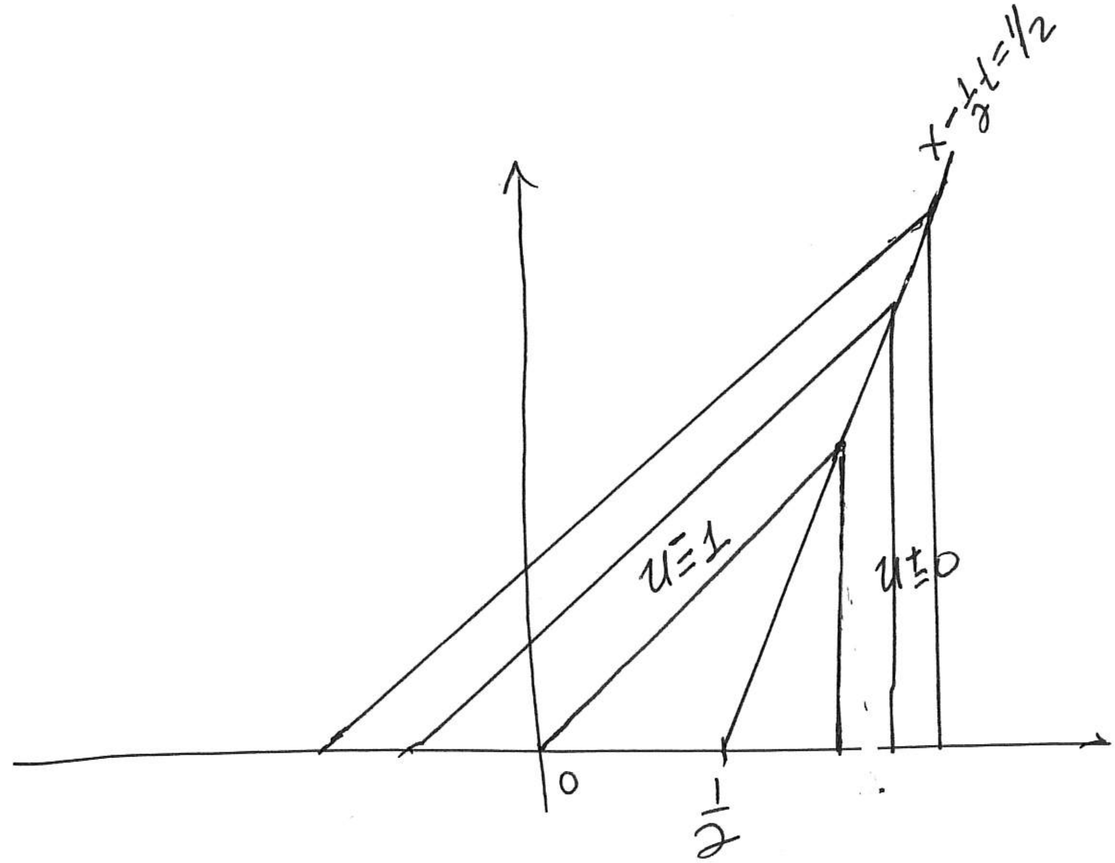
and $\xi(t) = \frac{1}{2}t + c$, $c=0$ b/c the shock passes by $(1,1)$

$c = 1/2$

So $\boxed{\xi(t) = \frac{1}{2}t + \frac{1}{2}}$



Wrong graph
See good one next page



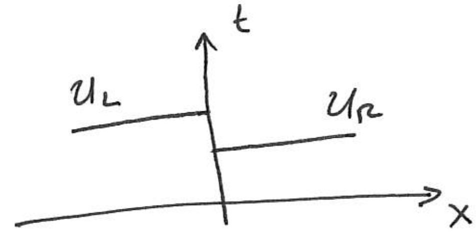
So

$$u(x,t)=\begin{cases}1, & x-t/2<1/2\\ 0, & x-t/2>1/2\end{cases}$$

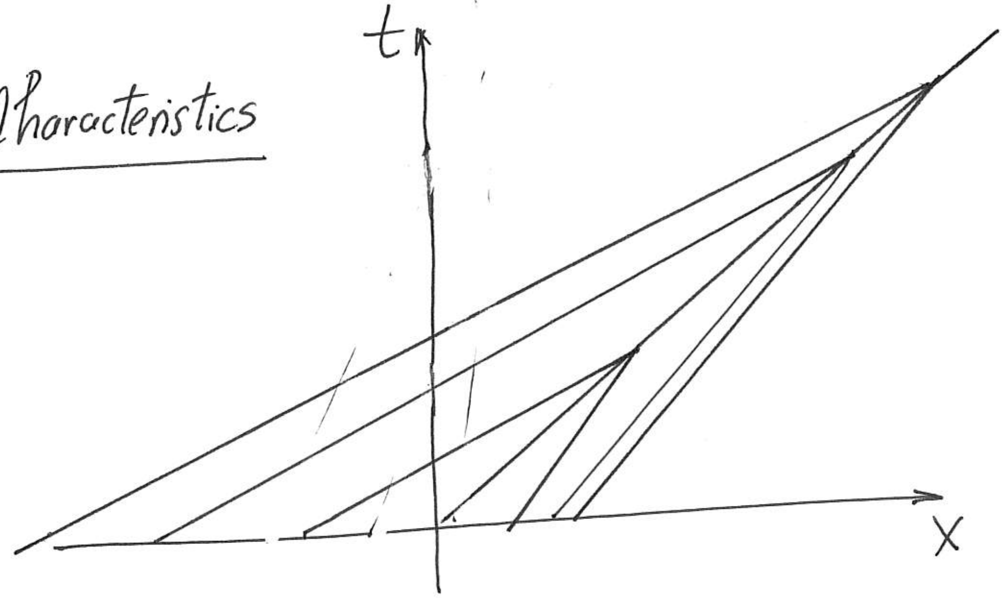
The Reimann Problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = f(x) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases} \end{cases}$$

Case 1: $u_L > u_R$



Characteristics



So, in this case

Characteristics: $x(t) = u(x_0,0)t + x_0 = f(x_0)t + x_0$

or $x(t) = \begin{cases} u_L t + x_0, & x_0 < 0 \\ u_R t + x_0, & x_0 > 0 \end{cases}$

Schock: $\dot{\xi}(t) = \dfrac{\frac{u_R^2}{2} - \frac{u_L^2}{2}}{u_R - u_L} = \dfrac{1}{2}(u_R + u_L)$

$f(u) = \dfrac{u^2}{2}$

$$\Rightarrow \boxed{\xi(t) = \left(\frac{u_R + u_L}{2}\right)t.}$$